Dynamical critical first-passage percolation in two dimensions

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- Question: Does there exist an infinite 0-cluster (an infinite connected component that consists only of vertices with state 0)?

Phase transition

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  - if $p < p_c$, no infinite 0-cluster a.s.;
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- Facts:
  - Essentially due to Kesten: $p_c = 1/2$ on $\mathbb{T}$.
  - When $p = p_c = 1/2$, no infinite 0-cluster a.s.

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- Garban–Pete–Schramm: this set has Hausdorff dimension $31/36$ a.s.

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- Reformulation: Define the first-passage time to infinity
$$\rho = \inf\{T(\gamma) : \gamma \text{ is an infinite path starting from } 0\}.$$
Is $\rho < \infty$ a.s.?
- Equivalent to the original problem. Hence no if $p \leq 1/2$, yes if $p > 1/2$.

A generalization

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- $F(0) = 1/2$?

A sharper phase transition

Theorem (Damron-L.-Wang, '17, simplified)

If $F(0) = 1/2$, *then* $\rho < \infty$ *a.s.* $\iff \sum_{k=2}^{\infty} F^{-1}(1/2 + 2^{-k}) < \infty$.

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- Intuition: $F^{-1}(1/2 + 2^{-k})$ smaller $\implies$ the probability that $\tau_v \approx 0$ is higher $\implies$ easier to get to infinity in finite time.

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- Define $\rho_t = \rho$ at time t.
- Are there exceptional times?
- Two cases:
  - Case 1: $\sum_k F^{-1}(1/2 + 2^{-k}) = \infty$.

    $$\text{Exceptional times: } \{t \geq 0 : \rho_t < \infty\}.$$

  - Case 2: $\sum_k F^{-1}(1/2 + 2^{-k}) < \infty$.

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Case 1: Hausdorff dimension

Write $a_k = F^{-1}(1/2 + 2^{-k})$.

Theorem (Damron-Hanson-Harper-L., '21)

Assume $\sum_k a_k = \infty$. *Then*

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- Generalizes the result of Garban–Pete–Schramm.

Case 1: Upper Minkowski dimension

Theorem (Damron-Hanson-Harper-L., '21)

Assume $\sum_k a_k = \infty$.

- *If* $ka_k \to \infty$, *then for any* $x \in [0,\infty)$,

$$\lim_{s\to\infty} \mathbf{P}\left(\overline{\dim}_{\mathrm{M}}(\{t \in [0,s] : \rho_t \le x\}) = \frac{31}{36}\right) = 1.$$

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- *If* $\liminf_k ka_k = 0$, *then for any* $x \in (0,\infty)$,

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- A further phase transition!

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- Can show: there exists distribution such that
  1. the set of exceptional times has different upper and lower Minkowski dimensions,
  2. and the upper Minkowski dimension of $\{t \geq 0 : \rho_t \leq x\}$ depends on x.

Case 2

Theorem (Damron-Hanson-Harper-L., '21)

Assume $\sum_k a_k < \infty$. *If further* $\sum_k k^{7/8} a_k < \infty$ *then a.s.*

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Idea of proofs

- See screen/board.

Thank you!